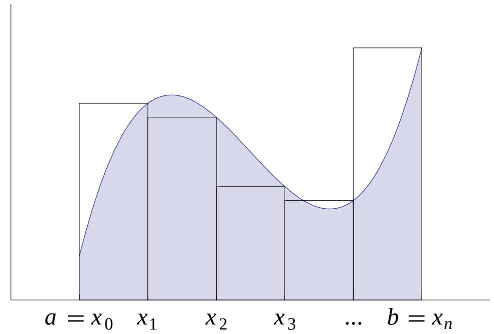
4.2 The Definite Integral

The **definite integral of a function** $f(x)$ **from** a **to** b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

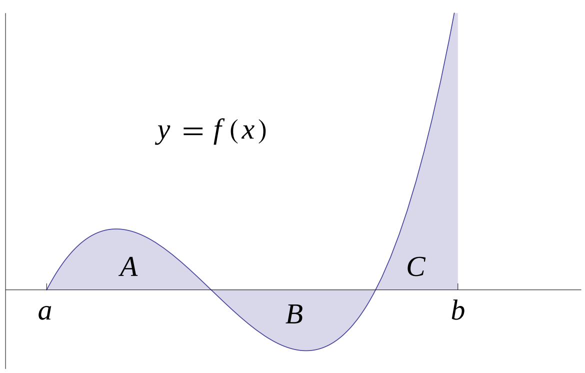


where $\Delta x = (b-a)/n$ is the width of the subintervals and $x_i = a + i\Delta x$, $i = 1, \ldots, n$ are equally spaced points from a to b.

The symbol $\int$ is called an **integral sign**, the function $f(x)$ is called the **integrand**, a is the **lower limit**, and b is the **upper limit** of the integral. The symbol dx has no intrinsic meaning, but reflects Δx in the limit and specifies the variable.

If the above limit exits, we say that $f(x)$ is **integrable**. Any function that is continuous or has only a finite number of jump discontinuities on $[a, b]$ is integrable. The process of finding $\int_a^b f(x)\,dx$ is called **integration**. The sum $\sum_{i=1}^{n} f(x_i)\Delta x$ is called a **Riemann Sum**. We are using the right endpoints of the subintervals in this definition, but any **sample point** x_i^* in $[x_{i-1}, x_i]$ may be used instead.

The definition is the same as our earlier definition of area under a graph, except that we do not assume that $f(x)$ is positive. The result is still connected to area. The definite integral can be viewed as the sum of the areas under $y = f(x)$ where $f(x) > 0$ minus the sum of the areas under $y = f(x)$ where $f(x) < 0$.



$$\int_a^b f(x)\,dx = A - B + C$$

Properties of the Integral

1. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ 2. $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$ 4. $\displaystyle f(x) \le g(x) \Rightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$

A special case of 1, is $\displaystyle\int_a^b c\,dx = c\int_a^b 1\,dx = c(b-a)$. Also, from 4, if $m \le f(x) \le M$ then $\displaystyle m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

In addition to these rules, we adopt the conventions

$$\int_a^a f(x)\,dx = 0, \qquad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

which are compatible with all the other rules and allow more freedom in manipulating integrals.

Example: Estimate $\displaystyle\int_1^8 \sqrt[3]{x}\,dx$

Solution: Since $1 = \sqrt[3]{1} \le \sqrt[3]{x} \le \sqrt[3]{8} = 2$, the above rules imply $\displaystyle 1(8-1) = 7 \le \int_1^8 \sqrt[3]{x}\,dx \le 2(8-1) = 14$. We can better estimate the value of the integral by using the average, $\displaystyle\int_1^8 \sqrt[3]{x}\,dx \approx \frac{7+14}{2} = 10.5$.

Of course, we may use Riemann Sums to estimate the integral as well. If we use $n = 7$ subintervals, then $\Delta x = (8-1)/7 = 1$ and $x_i = 1 + i$, for $i = 1, \ldots, 7$, so

$$\begin{aligned}\int_1^8 \sqrt[3]{x}\,dx &\approx \sum_{i=1}^{7} f(x_i)\Delta x = f(2)+f(3)+f(4)+f(5)+f(6)+f(7)+f(8)\\ &= \sqrt[3]{2}+\sqrt[3]{3}+\sqrt[3]{4}+\sqrt[3]{5}+\sqrt[3]{6}+\sqrt[3]{7}+\sqrt[3]{8}\\ &= 11.73\end{aligned}$$

If we use $n = 70$, then $\Delta x = (8-1)/70 = 1/10 = 0.1$ and $x_i = 1 + i/10$, so

$$\begin{aligned}\int_1^8 \sqrt[3]{x}\,dx &\approx \sum_{i=1}^{70} f(x_i)\Delta x\\ &= 0.1\Big(f(1.1)+f(1.2)+f(1.3)+\cdots+f(2.1)+f(2.2)+f(2.3)+\cdots+f(8.0)\Big)\\ &= 0.1\Big(\sqrt[3]{1.1}+\sqrt[3]{1.2}+\sqrt[3]{1.3}+\cdots+\sqrt[3]{2.1}+\sqrt[3]{2.2}+\sqrt[3]{2.3}+\cdots+\sqrt[3]{8.0}\Big)\\ &= 11.29979 \qquad \text{(The exact value of the integral is 11.25.)}\end{aligned}$$

Example: Approximate $\displaystyle\int_1^3 \frac{1}{x}\,dx$ using a Riemann Sum with $n = 5$. Take the sample points to be the midpoints of the subintervals.

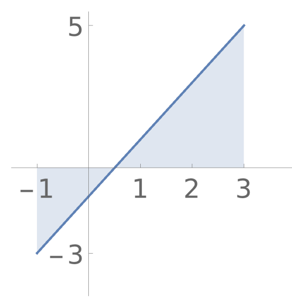
Solution: Since $\Delta x = (3-1)/5 = 0.4$, the endpoints of the 5 subintervals are $x_0 = 1$, $x_1 = 1.4$, $x_2 = 1.8$, $x_3 = 2.2$, $x_4 = 2.6$, $x_5 = 3.0$. The midpoints are thus $x_1^* = 1.2$, $x_2^* = 1.6$, $x_3^* = 2.0$, $x_4^* = 2.4$,

$x_5^* = 2.8$. The corresponding Riemann Sum is

$$\begin{aligned} \int_1^3 \frac{1}{x}\,dx &\approx \sum_{i=1}^{5} f(x_i^*)\Delta x \\ &= 0.4\Big(f(1.2)+f(1.6)+f(2.0)+f(2.4)+f(2.8)\Big) \\ &= 0.4\Big(\frac{1}{1.2}+\frac{1}{1.6}+\frac{1}{2.0}+\frac{1}{2.4}+\frac{1}{2.8}\Big) \\ &= 1.092857 \end{aligned}$$

(The exact value of the integral is 1.098612.)

Example: Use the definition of the integral to calculate $\int_{-1}^{3} 2x - 1\,dx$.



Solution: The region under the graph consists of two triangles one above the x-axis with area $\frac{1}{2}\cdot\frac{5}{2}\cdot 5 = \frac{25}{4}$, and one below the x-axis with area $\frac{1}{2}\cdot\frac{3}{2}\cdot 3 = \frac{9}{4}$, so we know the answer must be $\int_{-1}^{3} 2x-1\,dx = \frac{25}{4}-\frac{9}{4} = 4$. To use the definition, we work out a general formula for the Riemann Sum: $\Delta x = \frac{3-(-1)}{n} = \frac{4}{n}$, $x_i = a + i\Delta x = -1 + i\frac{4}{n}$, and

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\Big(2\Big(-1+i\frac{4}{n}\Big)-1\Big)\frac{4}{n} = -\frac{12}{n}\sum_{i=1}^{n}1 + \frac{32}{n^2}\sum_{i=1}^{n} i = -12 + \frac{32n(n+1)}{2n^2} = 4 + \frac{16}{n}$$

Here we are using a well-known formula $\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$. The value of the integral is the limit of the Riemann Sum as $n \to \infty$ which is 4, as expected.